JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 26, No. 3, August 2013

SEPARABLE REFLEXIVE BANACH SUBLATTICES OF $WeakL^1$

JeongHeung Kang*

Abstract. We investigate complemented Banach sublattices of the Banach envelope of $WeakL^1$. In particular, the Banach envelope of $WeakL^1$ contains a complemented Banach sublattice that is isometrically isomorphic to a separable reflexive Banach lattice.

1. Introduction

The space $WeakL^1$ was introduced in analysis when it was observed that some important operators such as the Hilbert transform and the Hardy-Littlewood maximal functions did not map L^1 into L^1. In this view point, it became natural to investigate $WeakL^1$, the space of measurable functions f satisfying

$$\mu(\{x \in \Omega : |f(x)| > y\}) < \frac{c}{y}. \tag{1.1}$$

For $0 < p < \infty$, the space $WeakL^p$ taken over the measure space (Ω, Σ, μ) consists of all equivalence classes of measurable functions f for which the quasinorm

$$q_p(f) = \sup_{a>0} a[\mu(\{x \in \Omega : |f(x)| > a\})]^{\frac{1}{p}} \tag{1.2}$$

is finite. Define q to be the Minkowski functional of the convex hull of the unit ball $\{f \in WeakL^1 : q_1(f) \leq 1\}$ of $WeakL^1$, where

$$q_1(f) = \sup_{a>0} a\mu(\{x \in \Omega : |f(x)| > a\})$$

Received April 16, 2013; Accepted July 16, 2013.
2010 *Mathematics Subject Classifications*: Primary 46B03.
Key words and phrases: complemented Banach sublattices, Banach envelope of $WeakL^1$, reflexive Banach sublattice.
This paper was partially supported by the Hwa Rang Dae Institute in 2012.

It was shown in [2] that

$$q(f)=\inf_{f=f_1+\cdots+f_n}\sum_{i=1}^{n}q_1(f_i), \tag{1.3}$$

where the infimum is taken over all finite decompositions $f=f_1+\cdots+f_n$ of f in $WeakL^1$. So q is a seminorm on $WeakL^1$. The quotient space of $WeakL^1$ modulo the subspace of elements f satisfying $q(f)=0$ is a Banach space normed by q whose dual coincides with the dual of $WeakL^1$. In [2], M. Cwikel and C. Fefferman showed that if μ is nonatomic then we get an equivalent integral-like seminorm

$$\|f\|_{wL_{\hat{1}}}=\lim_{n\to\infty}\sup_{\frac{q}{p}\geq n}\frac{1}{\ln\frac{q}{p}}\int_{\{p\leq|f|\leq q\}}|f|d\mu. \tag{1.4}$$

It was shown in [3] that the Banach envelope seminorm on $WeakL^1$ is exactly the same as the seminorm on $WeakL^1$ given in (1.4). Note that the seminorm on $WeakL^1$ given in (1.4) is a lattice seminorm. This is not quite obvious, but using integration by parts one can show (see [7, 1.5]) that $\|f\|_{wL_{\hat{1}}}$is exactly the same as

$$\lim_{n\to\infty}\sup_{\frac{q}{p}\geq n}\frac{1}{\ln\frac{q}{p}}\int_{p}^{q}\mu(\{x\in\mu:|f(x)|>t\})dt. \tag{1.5}$$

Even though $WeakL^1$ is complete with respect to the quasinorm q_1, it is not complete with respect to the seminorm $\|\cdot\|_{wL_{\hat{1}}}$. This is due to M. Cwikel and C. Feffreman in [3] and also we can see this in [7, 1.4].

Now, let $\mathcal{N}=\{f\in WeakL^1:\|f\|_{wL_{\hat{1}}}=0\}$. Then we obtain the quotient space $WeakL^1/\mathcal{N}$. We define $wL_{\hat{1}}$ as the *normed envelope* (and its completion as the *Banach envelope)* of $WeakL^1$.

It is known that $WeakL^1$ is not normable except for some trivial measure spaces (see [4]). In [4, Theorem 6], the authers showed that there exist nontrivial continuous linear functionals on $WeakL^1$. This implies that $WeakL^1$ has a nontrivial dual space. Moreover, in [7], J. Kupka and T. Peck showed that the space L^∞ is dense in the dual space of $WeakL^1$ with weak*-topology, and that there exist lattice embeddings of L^1, $l^1[0,1]$, l^∞ and $c_0[0,1]$ into $wL_{\hat{1}}$. In particular, the author proved that there exists a lattice isometry $T : L^1 \to wL_{\hat{1}}$ whose range is a complemented subspace of $wL_{\hat{1}}$ (see [7, Theorem 3.9]. Later on, T. Peck and M. Talagrand [10, Theorem 1] proved that every separable order continuous Banach lattice is lattice isometric to a sublattice of $wL_{\hat{1}}$ and H. Lotz and T. Peck removed the hypothesis 'order continuity' (see [9, Theorem 2]).

In this paper, we will show that the Banach envelope $wL_{\hat{1}}$ of $WeakL^1$ contains a lot of complemented Banach sublattices. In particular, for a separable reflexive Banach lattice E, we find a lattice isometry $T : E \to wL_{\hat{1}}$ such that the range of T is a complemented sublattice of the Banach envelope $wL_{\hat{1}}$ of $WeakL^1$. So the main result of this paper is an extension of [7, Theorem 3.9]. Since for $1 < p < \infty$, L^p space is a reflexive Banach lattice, we can see that L^p is a complemented sublattice of $wL_{\hat{1}}$. We will give the answer for this at the end of section 2.

2. Separable complemented sublattice in $wL_{\hat{1}}$.

To study this subject, we need some basic facts about the dual of $wL_{\hat{1}}$. We would like to change nonlinear limit superior expression (1.4) for $\|\cdot\|_{wL_{\hat{1}}}$ into a linear expression by directing the number $I_a^b(f) = \frac{1}{\ln \frac{b}{a}} \int_{\{a \le |f| \le b\}} |f| d\mu$ in some fashion. By [4, §1], we can define (1.4) as

$$\|f\|_{wL_{\hat{1}}} = \lim_{n\to\infty} (\sup\{I_a^b(f) : b/a \ge n\}). \tag{2.1}$$

For this, we introduce an ultrafilter $\mathcal{U}$ so that the limit of the I_a^b along $\mathcal{U}$ will determine a canonical integral-like linear functional $I_{\mathcal{U}} \in wL_{\hat{1}}^*$.

We now construct an ultrafilter $\mathcal{U}$ (see [7, §2.1]). For $n = 1, 2, \cdots$, let $F_n = \{(a,b) : 1 \le a < b, \frac{b}{a} \ge n\}$ and then define $\mathcal{F} = \{F_n : n \ge 1\}$. Treating $\mathcal{F}$ as a filter of subsets of the set $S = [1,\infty) \times [1,\infty)$, we obtain from Zorn's lemma an ultrafilter $\mathcal{U}$ of subsets of S such that $\mathcal{F} \subset \mathcal{U}$. From now, we'll fix the ultrafilter $\mathcal{F} \subset \mathcal{U}$. The significance of the ultrafilter property lies in the fact that for every function $f \in WeakL^1$, and for every integer n sufficiently large $\{I_a^b(f) : (a,b) \in F_n\}$ is bounded, so that the limit $l = \lim_{\mathcal{U}} I_a^b(f)$

always exists(for every $\epsilon > 0$, there is a set $U \in \mathcal{U}$ such that $|I_a^b(f) - l| < \epsilon$ whenever $(a, b) \in U$).

Define the "ersatz integral" $I_{\mathcal{U}}$ for every nonnegative function $f \in wL_{\hat{1}}$ by $I_{\mathcal{U}}(f) = \lim_{\mathcal{U}} I_a^b(f)$. For more properties of $I_{\mathcal{U}}(f)$, refer to [7, 2.3 key lemma]. We define for an arbitrary function $f \in wL_{\hat{1}}$ by $I_{\mathcal{U}}(f) = I_{\mathcal{U}}(f^+) - I_{\mathcal{U}}(f^-)$. Then we have $|I_{\mathcal{U}}(f)| \leq \|f\|_{wL_{\hat{1}}}$. Define $\|f\|_{\mathcal{U}} = I_{\mathcal{U}}(|f|)$. Note that (see [7, 2.12])

$$\|f\|_{\mathcal{U}} \leq \|f\|_{wL_{\hat{1}}}. \tag{2.2}$$

For the dual of $WeakL^1$(or $wL_{\hat{1}}$), we state the theorem which is due to J. Kupka and T. Peck in [7, 2.8].

Theorem 2.1. *Define a linear operator $T_{\mathcal{U}} : L_\infty(\mu) \to {WeakL^1}^*$ by $T_{\mathcal{U}}(m) = I_{\mathcal{U}}(mf)$ for all $m \in L_\infty$, and for all $f \in WeakL^1$. Then $T_{\mathcal{U}}$ constitutes an isometric order isomorphism of $L_\infty(\mu)$ into ${WeakL^1}^*$. Moreover, the linear span of the subspaces $T_{\mathcal{U}}(L_\infty(\mu))$, as $\mathcal{U}$ ranges over the collection of ultrafilter (of subset of S) which contain $\mathcal{F}$ constitutes a norming, and hence a weak* dense, subspace of ${WeakL^1}^*$.*

The operator $T_{\mathcal{U}}$ of Theorem 2.1 determines an isometric order isomorphic embedding of $L_\infty(\mu)$ into $WeakL^1(\mathcal{U})^*$ where $WeakL^1(\mathcal{U}) = \{f \in WeakL^1 : \|f\|_{\mathcal{U}} < \infty\}$. Moreover, the range of this embedding is norming, and hence $weak^*$ dense in $WeakL^1(\mathcal{U})^*$.

Let $L(\mathcal{U}) = \{f \in WeakL^1 : \|f\|_{wL_{\hat{1}}} = \|f\|_{\mathcal{U}}\}$. Then $L(\mathcal{U})$ is a closed subset of $wL_{\hat{1}}$(see [6]) and if f is a $\frac{1}{x}$-like function, then $\|f\|_{wL_{\hat{1}}} = \|f\|_{\mathcal{U}} = I_{\mathcal{U}}(f)$.

Lemma 2.2. *If $\phi \neq 0$ is a linear functional on $WeakL^1(\mathcal{U})$, then ϕ is a linear functional on $wL_{\hat{1}}$ with $\|\phi\| \neq 0$.*

Proof. Let $\phi \neq 0$ be a linear functional on $WeakL^1(\mathcal{U})$. Then for any $f \in wL_{\hat{1}}$ with $\|f\|_{\mathcal{U}} > 0$ (since $f \in wL_{\hat{1}}$ is also regarded as $f \in WeakL^1(\mathcal{U})$).

$$\begin{aligned} 0 < |\phi(f)| &\leq \|\phi\|\|f\|_{\mathcal{U}} \\ &\leq \|\phi\|\|f\|_{wL_{\hat{1}}} \quad \text{by (2.2).} \end{aligned}$$

Hence, $\|\phi\| \neq 0$ on $wL_{\hat{1}}$. This implies $\phi \neq 0$ is a linear functional on $wL_{\hat{1}}$.□

We now give a lemma about linear functionals on $wL_{\hat{1}}$ which is actually due to J. Kupka and T. Peck (see [7, 2.20]).

LEMMA 2.3. *For a ultrafilter $\mathcal{U}$ defined as above, let $f \in wL_{\hat{1}}$ be a nonnegative function with $\|f\|_{\mathcal{U}} = 1$. Then for any $g \in wL_{\hat{1}}$, disjointly supported from f, we can find $\phi \in wL_{\hat{1}}^*$ such that $\|\phi\| = 1$, $\phi(f) = 1$ and $\phi(g) = 0$.*

Let $(f_n)_{n=1}^{\infty}$ be a sequence of nonnegative elements in $wL_{\hat{1}}$ with $\|f_n\|_{wL_{\hat{1}}} = 1$, for all $n = 1, 2, 3, \cdots$ and such that the f_n have pairwise disjoint supports. Applying the inductive argument to Lemma 2.3, for each f_n, we can find a linear functional ϕ_n on $wL_{\hat{1}}$ such that $\phi_n(f_n) = 1$, $\|\phi_n\| = 1$ and $\phi_n(f_m) = 0$ if $n \neq m$.

LEMMA 2.4. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of nonnegative elements in $wL_{\hat{1}}$ such that the f_n have pairwise disjoint supports with $\|f_n\|_{wL_{\hat{1}}} = 1$, for all $n = 1, 2, \cdots$ and let $(\phi_n)_{n=1}^{\infty}$ be a sequence of linear functionals on $wL_{\hat{1}}$ selected as above. Then for any $f \in wL_{\hat{1}}$, we have $\sum_{n=1}^{\infty} |\phi_n(f)| \leq \|f\|_{wL_{\hat{1}}}$.*

Proof. For an arbitrary element $f \in wL_{\hat{1}}$, the number $\phi_n(f)$ is the limit of a subnet of the sequence $\{I_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)\}$ where $(E_{n,k})_{k=1}^{\infty}$ is a decreasing sequence of subsets of $E_n = supp(f_n)$, and f_n is bounded on $E_{n,k}^c$ for all k (see [7, 2.20]). Fix $n \neq m$, let $(E_{n,k})_{k=1}^{\infty}$ be the decreasing sequence of measurable sets for f_n and $(E_{m,k})_{k=1}^{\infty}$ the corresponding sequence for f_m. Let $r = sgnI_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)$, $s = sgnI_{\mathcal{U}}(\chi_{E_{m,k}} \cdot f)$. Put $m = r\chi_{E_{n,k}} + s\chi_{E_{m,k}}$ so that $\|m\|_{\infty} = 1$. By Theorem 2.1 and Lemma 2.3, we can identify $T_{\mathcal{U}}(m) = \widehat{m}$ as a linear functional on $wL_{\hat{1}}$. Then we have

$$\begin{aligned}
\widehat{m}(f) &= |I_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)| + |I_{\mathcal{U}}(\chi_{E_{m,k}} \cdot f)| \\
&= I_{\mathcal{U}}(m \cdot f) \\
&\leq \|m\|_{\infty} \|f\|_{\mathcal{U}} \quad \text{since } \|m\|_{\infty} = 1 \\
&= \|f\|_{\mathcal{U}} \quad \text{by (2.2)} \\
&\leq \|f\|_{wL_{\hat{1}}}.
\end{aligned}$$

By the additive rule for nets [5, Lemma 6, p28], we can say that in the limit

$$\begin{aligned}|\phi_n(f)| + |\phi_m(f)| &\le \|f\|_{\mathcal{U}} \quad \text{by (2.2)}\\ &\le \|f\|_{wL_{\hat{1}}}.\end{aligned}$$

To show $\sum_{n=1}^{\infty} |\phi_n(f)| \le \|f\|_{wL_{\hat{1}}}$, it suffices to show that for any $N \in \mathbf{N}$, $\sum_{n=1}^{N} |\phi_n(f)| \le \|f\|_{wL_{\hat{1}}}$. For $n = 1, 2, \cdots$, let $(E_{n,k})_{k=1}^{\infty}$ be the decreasing sequence of measurable sets for f_n and $E_n = supp(f_n)$. Let $r_n = sgn(\chi_{E_{n,k}} \cdot f)$. Put $m = \sum_{n=1}^{N} r_n \chi_{E_{n,k}}$. Then we have $\|m\|_\infty = 1$. By the same argument as above, one can get

$$\begin{aligned}\widehat{m}(f) &= \sum_{n=1}^{N} |I_{\mathcal{U}}(\chi_{E_{n,k}} \cdot f)|\\ &= I_{\mathcal{U}}(m \cdot f)\\ &\le \|m\|_\infty \|f\|_{\mathcal{U}} \quad \text{since } \|m\|_\infty = 1 \text{ and by (2.2)}\\ &\le \|f\|_{wL_{\hat{1}}}.\end{aligned}$$

By the additive rule for nets [5, Lemma 6, p28], we can say that in the limit

$$\begin{aligned}\sum_{n=1}^{N} |\phi_n(f)| &\le \|f\|_{\mathcal{U}} \qquad \text{since } \|m\|_\infty = 1\\ &\le \|f\|_{wL_{\hat{1}}}.\end{aligned}$$

We can therefore say that $\sum_{n=1}^{\infty} |\phi_n(f)| \le \|f\|_{wL_{\hat{1}}}$.
This proves the lemma. □

We now need to recall the T. Peck and M. Talagrand's theorem. In [10, Theorem 1], one can see the following theorem; Let Ω be a set and $\Omega_{i,n}$, $n \ge 0$, $1 \le i \le 2^n$ be a set of Ω such that $\Omega_{1,0} = \Omega$, $\Omega_{i,n} \cap \Omega_{j,n} = \emptyset$, if $i \ne j$ and $\Omega_{i,n} = \Omega_{2i-1,n+1} \cup \Omega_{2i,n+1}$. Let $\chi_{i,n}$ be the characteristic function of $\Omega_{i,n}$, $n > 0$, $1 \le i \le 2^n$ and let Y be the linear span of the functions $\chi_{i,n}$, $n > 0$, $1 \le i \le 2^n$.

Theorem 2.5. [10, T. Peck and M. Talagrand] *Let X be the completion of Y under some lattice norm on Y where Y is given the usual pointwise order. Then there is a lattice isometry of X into $wL_{\hat{1}}$.*

T. Peck and M. Talagrand constructed for all natural number n, $1 \le i \le 2^n$ under lattice isometry T, $T\chi_{i,n} = f_{i,n}$, where

$$f_{i,n} = \sum_{m \ge n} \sum_{j=1}^{2^{m-n}} e_{2^{m-n}(i-1)+j,m}$$

and for $x \in [v_{i,n}, w_{i,n}]$, each $e_{i,n}(x) = \frac{b_{i,n}}{x-u_{i,n}}$ is a $\frac{1}{x}$-like function. Note that $f_{i,n}$ are all nonnegative and pairwise disjointly supported in $wL_{\hat{1}}$ and $f_{i,n} = f_{2i,n+1} + f_{2i+1,n+1}$, for all n, and $1 \le i \le 2^n$ (see [10, proof of Theorem 1]).

Theorem 2.6. *Let E be a separable reflexive Banach lattice and $T : E \longrightarrow wL_{\hat{1}}$ be the lattice isometry given in Theorem 2.5. Then the range of T is a complemented sublattice of $wL_{\hat{1}}$.*

Proof. Let E be a reflexive Banach lattice. Then TE is also a reflexive sublattice of $wL_{\hat{1}}$. This implies that the unit ball B_{TE} is weakly compact. Since every separable reflexive Banach lattice has an order continuous norm, E has an order continuous norm. Hence we can apply the construction of T in Theorem 2.5. Let $(\chi_{i,n})_{i=1}^{2^n}$ be the subset of E defined in Theorem 2.5. Without loss of generality, one can assume $\|\chi_{i,n}\| = 1$ for all $1 \le i \le 2^n$ by normalizing. Then we have $\overline{span}(\chi_{i,n})_{i=1}^{2^n} \subset E$. Define $T\chi_{i,n} = f_{i,n}$, then $\overline{span}(f_{i,n}) \simeq \overline{span}(\chi_{i,n})$. Since $\{\chi_{i,n}\}$ form a dense subset of E, $\{f_{i,n}\}$ form a dense subset of TE. Moreover, for fixed n, the $f_{i,n}$ are pairwise disjointly supported nonnegative elements in TE with $\|f_{i,n}\|_{wL_{\hat{1}}} = 1$. Hence by Lemma 2.3, we can find linear functionals $\phi_{i,n}$ on $wL_{\hat{1}}$ such that $\phi_{i,n}(f_{j,n}) = \delta_{i,j}$ and $\|\phi_{i,n}\| = 1$, for all $i = 1, 2, \cdots$. For each n, let $B_n = \{f_{i,n}\}_{i=1}^{2^n}$ and define $P_{B_n} : wL_{\hat{1}} \longrightarrow \overline{span}(f_{i,n})_{i=1}^{2^n} \subset TE$ by

$$P_{B_n}(f) = \sum_{i=1}^{2^n} \phi_{i,n}(f) f_{i,n}. \tag{2.3}$$

Since, for all $f \in wL_{\hat{1}}$, by Lemma 2.4 and $\|f_{i,n}\|_{wL_{\hat{1}}} = 1$

$$\|P_{B_n}(f)\|_{wL_{\hat{1}}} = \| \sum_{i=1}^{2^n} \phi_{i,n}(f) f_{i,n} \|_{wL_{\hat{1}}}$$

$$\leq \sum_{i=1}^{2^n} |\phi_{i,n}(f)| \|f_{i,n}\|_{wL_{\hat{1}}}$$

$$\leq \sum_{i=1}^{2^n} |\phi_{i,n}(f)| \leq \|f\|_{wL_{\hat{1}}}. \tag{2.4}$$

This implies $\|P_{B_n}\| \leq 1$, and P_{B_n} is a well defined linear map. Moreover, $f_{j,n} \in TE \subset wL_{\hat{1}}$,

$$P_{B_n}(f_{j,n}) = \sum_{i=1}^{2^n} \phi_{i,n}(f_{j,n}) f_{i,n}$$

$$= \phi_{j,n}(f_{j,n}) f_{j,n} = f_{j,n}. \tag{2.5}$$

Hence $\|P_{B_n}(f_{j,n})\|_{wL_{\hat{1}}} = \|f_{j,n}\|_{wL_{\hat{1}}} = 1$, and $P_{B_n}^2 = P_{B_n}$. Hence P_{B_n} is a projection $wL_{\hat{1}}$ onto $\overline{span}(f_{i,n})_{i=1}^{2^n} \subset TE$. From this, we want to find a projection P from $wL_{\hat{1}}$ onto TE. We define a partial order on $\{B_n\}_{n=1}^{\infty}$ by $B_n \prec B_m$ if $\overline{span}(f_{i,n}) \subset \overline{span}(f_{i,m})$. Then for each B_n, we have $\|P_{B_n}(f)\|_{wL_{\hat{1}}} \leq \|f\|_{wL_{\hat{1}}}$, for all $f \in wL_{\hat{1}}$ by (2.4). Hence the vector $P_{B_n}(f)$ belongs to $\{g \in TE : \|g\|_{wL_{\hat{1}}} \leq \|f\|_{wL_{\hat{1}}}\}$ which is a weakly compact subset in TE. Now consider the following product space;

$$\prod_{f \in wL_{\hat{1}}} \{g \in TE : \|g\|_{wL_{\hat{1}}} \leq \|f\|_{wL_{\hat{1}}}\}. \tag{2.6}$$

Note that by Tychonoff's theorem, $\prod_{f \in wL_{\hat{1}}} \{g \in TE : \|g\|_{wL_{\hat{1}}} \leq \|f\|_{wL_{\hat{1}}}\}$ is compact for the weak topology. Hence the net $\{P_{B_n}\}$ of projections from $wL_{\hat{1}}$ to TE has a subnet which converges to some limit point P, in the topology of pointwise convergence on $wL_{\hat{1}}$, taking the weak topology on TE. Let $\{P_{B_{n_\alpha}}\}$ be a subnet of $\{P_{B_n}\}$ which converges to P. Then we have the weak limit $P(f) = \lim_\alpha P_{B_{n_\alpha}}(f)$, for all $f \in wL_{\hat{1}}$. Since each P_{B_n} is contractive, positive, and norm one, P is contractive, positive, and norm one.

Finally, we need to show that for all $f \in TE$, $P(f) = f$. Since $(f_{i,n})$ are dense, given $\epsilon > 0$ one can find $B_n = \{f_{i,n}\}$ such that $\| \sum_{i=1}^{2^n} a_i f_{i,n} - f\|_{wL_{\hat{1}}} < \epsilon/2$ for some $(a_i)_{i=1}^{2^n}$. Let $g = \sum_{i=1}^{2^n} a_i f_{i,n}$. Then since $\|P(g) - g\|_{wL_{\hat{1}}} = 0$, we can have

$$
\begin{aligned}
\|P(f)-f\|_{wL_{\hat{1}}} &\le \|P(f)-P(g)\|_{wL_{\hat{1}}} + \|P(g)-g\|_{wL_{\hat{1}}} + \|g-f\|_{wL_{\hat{1}}} \\
&\le \|P(f-g)\|_{wL_{\hat{1}}} + \|g-f\|_{wL_{\hat{1}}} \\
&\le \|f-g\|_{wL_{\hat{1}}} + \|g-f\|_{wL_{\hat{1}}} \\
&< \epsilon.
\end{aligned}
$$

Hence $P(f) = f$ for all $f \in TE$. Therefore P is a positive norm one projection from $wL_{\hat{1}}$ onto TE. This proves the theorem. □

Now for $(1 < p < \infty)$, we can have the $L_p(\mu)$ space structure which is lattice isometric to a complemented in $wL_{\hat{1}}$.

Corollary 2.7. *For $1 < p < \infty$, the Banach envelope of $WeakL_1$ contains a complemented sublattice that is isometrically isomorphic to $L_p(\Omega, \Sigma, \mu)$ where μ is a separable probability measure.*

Proof. For $1 < p < \infty$, $L_p(\mu)$ is a reflexive separable Banach lattice. By Theorem 2.5, there exists a lattice isometry T from $L_p(\mu)$ into $wL_{\hat{1}}$. Then $TL_p(\mu)$ is a separable reflexive Banach sublattice of the Banach envelope of $WeakL^1$. Hence by Theorem 2.6, one can find a projection P from $wL_{\hat{1}}$ onto $TL_p(\mu)$. Since $L_p(\mu)$ is lattice isometric to $TL_p(\mu)$, $TL_p(\mu)$ is the desired sublattice. This proves the corollary. □

Corollary 2.8. *Let E be a separable reflexive Banach lattice. Then any ideal I of E is lattice isometric to a complemented sublattice of $wL_{\hat{1}}$.*

Proof. Let $T : E \longrightarrow wL_{\hat{1}}$ be the isometric order isomorphism of Theorem 2.5. Then by Theorem 2.6, TE is complemented in $wL_{\hat{1}}$. Let $P : wL_{\hat{1}} \longrightarrow TE$ be a projection and I be an any ideal of E. Then TI is an ideal of TE. Since E is order continuous, TE is also an order continuous sublattice of $wL_{\hat{1}}$. Hence by Ando's theorem [8, 1.a.11], TI is the range of a positive projection from TE. Let $P_1 : TE \longrightarrow TI$ be a such projection. Then $Q = P_1 \circ P$ is the desired projection from $wL_{\hat{1}}$ onto TE. This proves the corollary. □

References

[1] M. Cwikel and Y. Sagher, $L(p,\infty)^*$, Indiana Uni. Math. J. **21** (1972), 782-786.
[2] M. Cwikel, *On the conjugates of some function spaces*, Studia Math. **45** (1973), 49-55.
[3] M. Cwikel and C. Fefferman, *Maximal seminorm on $WeakL^1$*, Studia Math. **69** (1980), 149-154.
[4] M. Cwikel and C. Fefferman, *The canonical seminorm on $WeakL^1$*, Studia Math. **78** (1984), 275-278.
[5] N. Dunford and J. T. Schwartz, Linear Operator I : General Theory. Pure and Applied Mathematics, New York: Interscience **VII**, 1967.
[6] J. Kang, *Banach subspaces and envelope norm of $wL_{\hat{1}}$*, Bull. Korean. Math. Soc. **35** (1998), 409-420.
[7] J. Kupka and T. Peck, *The L_1-structure of $WeakL^1$*, Math. Ann. **269** (1984), 235-262.
[8] Lindenstrauss and Tzafriri, *Classical Banach Spaces II*, Springer-Verlag, New York, 1974.
[9] H. P. Lotz and T. Peck, *Sublattices of the Banach envelope of $WeakL^1$*, Proc. Amer. Math. Soc. **126** (1998), 75-84.
[10] T. Peck and M. Talagrand, *Banach sublattices of $WeakL^1$*, Israel. J. Math. **59** (1987), 257-271.

*
Department of Mathematics
Korea Military Academy
Seoul 139-799, Republic of Korea
E-mail: `jkang@kma.ac.kr`